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Research Article

The generalization of fixed point theorem in the S-Menger space with applications

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Abstract: In this paper, we proved common fixed-point theorems in the structue of S-Menger spaces [8]. Our results provide significant generalizations and extensions of the results presented by M. Bousselsal*et al.* [1].
Keywords: *S*-menger spaces, control functions fixed point theorems, integral equation

1. Introduction

The idea of probabilistic metric spaces, introduced by Karl Menger in 1942, presents a new way to understand distance. Over the years, several generalizations of metric spaces have been proposed. B. C. Dhage [2, 3] introduced generalized metric spaces and investigated their topological properties. Similarly, S. Gähler [4,5] developed the concept of 2-metric spaces, which provided a new perspective on distance functions. Later, Z. Mustafa and B. Sims [6] proposed *G*-metric spaces as another important extension. Furthermore, S. Sedghi and co-authors [7] introduced *S*-metric spaces, enriching the structure of generalized metric frameworks.

Recently, Krishna Kanta Sarkar et al. introduced the concept of *S*-Menger spaces in 2023 [8], which extends the theory of Menger-type spaces within the framework of *S*-metric structures. This development provides a more generalized setting for the study of fixed-point results and mapping behaviour.In addition, several researchers have contributed to fixed point theory in these generalized settings [1], highlighting their importance in both theoretical and applied mathematics. The continuous evolution of such structures offers powerful tools for addressing complex problems in analysis and related areas.

In the present paper, we extend and generalize the results of M. Bousselsal*et al.* [1]. We establish the existence and uniqueness of a common fixed point in *S*-Menger spaces by employing a control function ϕ along with suitable conditions on mappings. Illustrative examples are also provided to support the obtained results.

2. Preliminaries and the definition of *S*-Menger Spaces

We assume that the function $\phi: [0,1] \rightarrow [0,1]$ satisfying the following properties:
(P1)ϕ is strictly decreasing and left continuous.
(P2) $\phi(m)= 0$ if and only if $m = 1$
obviously, we obtain that $\lim_{m\rightarrow 1-} \phi(m) = \phi(1) = 0$.

Definition 2.1. **[8]***S*-Menger space is defined as the 3-tuple $(X, S, *)$. if X is a non-empty set, S is a function defined on$X^3$to the set of distribution function and $*$ is a continuous third order t-norm such that the following conditions are satisfied:

(i) $S_{(\alpha,\beta,\gamma)}(0) = 0$ for all $\alpha, \beta, \gamma \in X$
(ii) $S_{(\alpha,\alpha,\beta)}(t) < 1$ for $t > 0$ with $\alpha \neq \beta$,
(iii) $S_{(\alpha,\beta,\gamma)}(t) = 1$ for all $t > 0$, if and only if $\alpha = \beta = \gamma$.
(iv) $S_{(\alpha,\beta,\gamma)}(t) \geq [S_{(\alpha,\alpha,a)}(t_1) * S_{(\beta,\beta,a)}(t_2) * S_{(\gamma,\gamma,a)}(t_3)]$

Where $t = t_1 + t_2 + t_3$ and $t, t_1, t_2, t_3 > 0$ for all $\alpha, \beta, \gamma, a \in X$

For the definitions of convergence sequence, Cauchy sequence and symmetricity reader can refer[8].

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3. Main Results

Theorem 3.1. Let $(X,S,*)$ be a complete S-Menger space and assume that ϕ: $[0,1]\rightarrow[0,1]$ satisfying the foregoing properties (P_1) and (P_2). Furthermore, let χ be a function $(0,\infty)\rightarrow(0,1)$. Let A and B be maps that satisfy the following condition.

(i) $B(X)\subseteq A(X)$

(ii) A is continuous.

$\phi(S_{(B(\alpha),B(\alpha),B(\beta)}(t))\leq \chi(t)\phi(S_{(A(\alpha),A(\alpha),A(\beta)}(t)))$…………….. (3.1)

where $\alpha,\beta\in X$, and $t>0$ then A and B have a unique fixed point provided A and B commute.

Proof. Let α_0 be a point in X. By hypothesis (i), we can fixed α_1 such that $A\alpha_1=B\alpha_0$, by induction we can define a sequence $\{\alpha_n\}$ in X such that $A\alpha_n=B\alpha_{n-1}$. By induction again and by (3.1) we have

$$\phi\left(S_{(A\alpha_n,A\alpha_n,A\alpha_{n+1})}(t)\right)=\phi(S_{(B\alpha_{n-1},B\alpha_{n-1},B\alpha_n)}(t))$$
$$\leq\chi(t)\phi(S_{(A\alpha_{n-1},A\alpha_{n-1},A\alpha_n)}(t))$$
$$<\phi(S_{(A\alpha_{n-1},A\alpha_{n-1},A\alpha_n)}(t))\ldots\ldots\ldots\ldots\ldots \text{(3.2)}$$

Since ϕ is strictly decreasing, then

$$S_{(A\alpha_n,A\alpha_n,A\alpha_{n+1})}(t)> S_{(A\alpha_{n-1},A\alpha_{n-1},A\alpha_n)}(t)\ \ldots\ldots\ldots\ldots\ \text{(3.3)}$$

Setting $\beta_n(t)=S_{(A\alpha_n,A\alpha_n,A\alpha_{n+1})}(t)$. For (3.3), the sequence $\{\beta_n(t)\}$ is strictly increasing and bounded then $\beta_n(t)$ converges to $\beta(t)$ for all $t>0$.

Assume that $\beta(t)\in]0,1[$. Since $\beta_n(t)>\beta_{n-1}(t)$ for all $t>0$, then

$$\phi(\beta_n(t))\leq\chi(t)\phi(\beta_{n-1}(t))$$

for every $t>0$. Letting $\rightarrow\infty$, since ϕ is left continuous, we have

$$\phi(\beta(t))\leq\chi(t)\phi(\beta(t))<\phi(\beta(t))$$

for every $t>0$. which is a contradiction. Hence $\beta(t)=1$, that is the sequence $\{\beta_n(t)\}$ converges to 1 for every $t>0$. Next, we show that the sequence $\{A\{\alpha_n\}\}$ is a Cauchy sequence. Assume that it is not, then there exist $0<\varepsilon<1$ and two sequences $\{p(n)\}$ and $\{q(n)\}$ such that

$$p(n)>q(n)\geq n$$
$$S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{q(n)})}(t)\leq 1-\varepsilon$$
$$S_{(A\alpha_{p(n)-1},A\alpha_{p(n)-1},A\alpha_{q(n)-1})}(t)>1-\varepsilon$$
$$S_{(A\alpha_{p(n)-1},A\alpha_{p(n)-1},A\alpha_{q(n)})}(t)>1-\varepsilon\ \ldots\ldots\ldots\ldots\ldots\ldots \text{(3.4)}$$

for each $n\in N\cup\{0\}$, we get $\delta_n(t)=S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{q(n)})}(t)$ then we have

$$1-\varepsilon\geq\delta_n(t)=S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{q(n)})}(t)$$
$$\geq S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{p(n)-1})}\left(\frac{t}{3}\right)$$
$$*S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{p(n)-1})}\left(\frac{t}{3}\right)$$
$$*S_{(A\alpha_{q(n)},A\alpha_{q(n)},A\alpha_{p(n)-1})}\left(\frac{t}{3}\right)$$
$$\geq\beta_{p(n)-1}\left(\frac{t}{3}\right)*\beta_{p(n)-1}\left(\frac{t}{3}\right)*1-\varepsilon\qquad \text{(by 3.4)}\ \ldots\ldots\ldots\ldots\ldots\ldots \text{(3.5)}$$

Since $\beta_{p(n)-1}\left(\frac{t}{3}\right)\rightarrow 1$ as $n\rightarrow\infty$ for every $t>0$. Supposing that $n\rightarrow\infty$, we note that the sequence $\{\delta_n(t)\}$ converges to $1-\varepsilon$ for every $t>0$. Moreover by (3.1) we have

$$S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{q(n)})}(t)$$
$$\leq\chi(t)\phi(S_{(A\alpha_{p(n)-1},A\alpha_{p(n)-1},A\alpha_{q(n)-1})}(t)$$
$$<\phi(S_{(A\alpha_{p(n)-1},A\alpha_{p(n)-1},A\alpha_{q(n)-1})}(t)\ldots\ldots \text{(3.6)}$$

According to the monotonicity of ϕ, we know that

$$S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{q(n)})}(t)$$
$$>(S_{(A\alpha_{p(n)-1},A\alpha_{p(n)-1},A\alpha_{q(n)-1})}(t)$$

for each n. Thus, on the basis of formula (3.6) we can obtain.

$$1-\varepsilon\geq S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{q(n)})}(t)>(S_{(A\alpha_{p(n)-1},A\alpha_{p(n)-1},A\alpha_{q(n)-1})}(t)>1-\varepsilon\ \ldots.. \text{(3.7)}$$

Clearly, this leads to a contraction. Hence $\{A\alpha_n\}$ is a Cauchy sequence. By the completeness of X, $\{A\alpha_n\}$ converges to β, so $B\alpha_{n-1}=A\alpha_n$ tends to β. It can be seen that from (3.1) and the left continuous of ϕ that the continuity of A implies the continuity of B. So $B(A\alpha_n)\rightarrow B(\beta)$. However $BA(\alpha_n)\rightarrow AB(\alpha_n)$ by the commutativity of A and B. So $A(B(\alpha_n))$ converges to $A(\beta)$. Because the limit is unique $A(\beta)=B(\beta)$ So $A\big(A(\beta)\big)=A(B(\beta))$ by commutativity and

$$\phi\ (S_{(B(\beta),B(\beta),B(B(\beta))}(t)$$
$$\leq\chi(t)\phi\ (S_{(A(\beta),A(\beta),A(B(\beta))}(t)$$
$$\leq\ \chi(t)\phi\ (S_{(B(\beta),B(\beta),B(B(\beta))}(t)$$
$$<\phi\ (S_{(B(\beta),B(\beta),B(B(\beta))}(t)$$

then if $B(\beta)\neq B(B(\beta))$, we have a contraction hence, $B(\beta)=B(B(\beta))$.

Then $B(\beta)=B(B(\beta))=A\big(B(\beta)\big)$. So $B(\beta)$ is a common fixed point of A and B. Now we prove the uniqueness of the common fixed point of A and B. If β and γ are two common fixed points to A and B, and $\beta\neq\gamma$, then

$$\phi\ (S_{(\beta,\beta,\gamma)}(t)=(S_{(B\beta,B\beta,B\gamma)}(t)$$
$$\leq\chi(t)\phi(S_{(A\beta,A\beta,A\gamma)}(t)$$
$$<\phi(S_{(A\beta,A\beta,A\gamma)}(t)$$

$$= \phi\,(S_{(\beta,\beta,\gamma)}(t)$$

then $S_{(\beta,\beta,\gamma)}(t) > S_{(\beta,\beta,\gamma)}(t)$, which is a contradiction so $\beta = \gamma$.

Remark 3.2. If we choose $A = I$ in theorem (3.1), we obtain the following corollary it will generalize in the setting of *S*-Menger space.

Corollary 3.3. Let $(X, S, *)$ be a complete *S*-Menger Spaces and B a self-map of X and assume that $\phi: [0,1] \to [0,1]$ satisfy the foregoing properties (P_1) and (P_2).Furthermore ,letχbe a function from $(0, \infty) \to (0,1)$.Let B be a continuous map that satisfies the following conditions:

$$\phi\,(S_{(B(\alpha),B(\alpha),B(\beta)}(t) \le \chi(t)\phi(S_{(\alpha,\alpha,\beta)}(t)$$

where $\alpha, \beta \in X$ and $t > 0$, then B has a unique fixed point.

We now give an example that illustrate our main result.

Example 3.4. Let $X = \{\frac{1}{n}, n \in N\} \cup \{0\}$ with the *S*-Menger space,*S* is defined by

$$S_{(\alpha,\beta,\gamma)}(t) = \begin{cases} 0, & \text{if } t = 0 \\ \dfrac{t}{t + |\alpha - \beta| + |\beta - \gamma|}, & \text{if } t > 0, \alpha, \beta, \gamma \in X. \end{cases}$$

Clearly $S_{(\alpha,\beta,\gamma)}(*)$ is complete *S*-Menger space on X. Which is defined by $a * b * c = abc$. Define $B(\alpha) = \frac{\alpha}{6}$ and $A(\alpha) = \frac{\alpha}{2}$ on X. It is evident that $B(X) \subset A(X)$. Also, define the function $\chi: (0, \infty) \to (0,1)$ by $\chi(t) = \frac{2+\frac{1}{t}}{6+\frac{1}{t}}$ for $t > 0$,the function $\phi: [0,1] \to [0,1]$ defined by $\phi(t) = \frac{1-t}{1+t}$ satisfies the properties (P_1) and (P_2).

$$S_{(B\alpha,B\alpha,B\beta)}(t) = \frac{3t}{3t + |\alpha - \beta|} t > 0, \alpha, \beta \in X$$

$$\phi(S_{(B\alpha,B\alpha,B\beta)}(t)) = \frac{|\alpha - \beta|}{6t + |\alpha - \beta|} t > 0 \alpha, \beta \in X$$

$$\phi(S_{(A\alpha,A\alpha,A\beta)}(t)) = \frac{|\alpha - \beta|}{2t + |\alpha - \beta|} t > 0, \alpha, \beta \in X$$

Since $|\alpha - \beta| \le 1$ for $\alpha, \beta \in X$ then it is easy to see that

$$\phi(S_{(B\alpha,B\alpha,B\beta)}(t)) \le \chi(t)\phi(S_{(A\alpha,A\alpha,A\beta)}(t)$$

All the hypothesis of theorem (3.1) are satisfied and thus A and B have a unique common fixed point $\alpha = 0$.

Application: Let $\Upsilon = \{\Omega: [0,1[\to [0,1[, \Omega$ is a Lebesgue integrable mapping which is summable, non-negative and satisfies $\int_{1-\varepsilon}^{1} \Omega(t)dt > 0$ for each $0 < \varepsilon < 1\}$

Theorem 3.5. Let $(X, S, *)$ be a complete *S*-Menger space and B a self-map of X and assume that $\phi: [0,1] \to [0,1]$ satisfies the foregoing properties (P_1) and (P_2).If for any $t > 0$, A and B satisfy the following condition:

$$\int_{1-\phi(S_{(B\alpha,B\alpha,B\beta)}(t))}^{1} \Omega(s)ds \le \chi(t) \int_{1-\phi(S_{(A\alpha,A\alpha,A\beta)}(t)}^{1} \Omega(s)ds \quad , \text{ for } \Omega \in \Upsilon \ldots \ldots \text{(3.5)}$$

where $\alpha, \beta \in X$, then A and B havea unique common fixed point.

Proof. For $\Omega \in \Upsilon$, we consider the function:

$\wedge: [0,1] \to [0,1]$ by $\wedge(\varepsilon) = \int_{1-\varepsilon}^{1} \Omega(s)ds$.

$\wedge$ is continuous, $\wedge(0) = 0$,$\wedge$ is strictly increasing (3.1) becomes.

$$\wedge\left(\phi\left(S_{(B\alpha,B\alpha,B\beta)}(t)\right)\right) \le \chi(t) \wedge (\phi(S_{(A\alpha,A\alpha,A\beta)}(t))$$

Setting $\phi_1 = \wedge o \phi$ and ϕ_1 is strictly decreasing, left continuous and satisfies the properties (P_1) and $(P_2)$for any $t > 0$, then by theorem (3.1), A and B have a unique common fixed point.

4. The Second Main Result

In this section, we assume that the functions $\phi, \Psi: [0,1] \to [0,1]$ satisfying properties:

(q_1) ϕ is strictly decreasing and left continuous,

(q_2) $\phi(m)$ = 0 if and only if *m*= 1

(q_3) $\Psi$is lower semi-continuous and $\Psi(m)$ = 0 if and only if m = 1. Obviously, we obtain that $\lim_{m \to 1-} \phi(m) = \phi(1) = 0$

Theorem 4.1. Let $(X, S, *)$ be a complete *S*-Menger Spaces and assume that $\phi, \Psi: [0,1] \to [0,1]$ satisfies the foregoing properties (q_1), $(q_2)$and (q_3). Let A and Bbemaps that satisfy the following condition:

(i) $B(X) \subset A(X)$

$(ii)$$A$ is continuous.

$$\phi(S_{(B(\alpha),B(\alpha),B(\beta)}(t)) \le \phi(S_{(A(\alpha),A(\alpha),A(\beta)}(t)) - \Psi(S_{(A(\alpha),A(\alpha),A(\beta)}(t)) \ldots\ldots\ldots\ldots\ldots\text{(4.1)}$$

where$\alpha, \beta \in X$ and $t > 0$, then A and B have a unique common fixed point provided A and B commute.

Proof. Let α_0 be a point in X. By hypothesis (i), we can fixed$\alpha_1 \in X$ such that $A\alpha_1 = B\alpha_0$, by induction, we can define a sequence $\{\alpha_n\}$ in X such that$A\alpha_n = B\alpha_{n-1}$. By induction again and by (4.1) we have

$$\phi(S_{(A(\alpha_n),A(\alpha_n),A(\alpha_{n+1})}(t)) = \phi(S_{(B(\alpha_{n-1}),B(\alpha_{n-1}),B(\alpha_n)}(t))$$

$$\le \phi(S_{(A(\alpha_{n-1}),A(\alpha_{n-1}),A(\alpha_n)}(t)) - \Psi(S_{(A(\alpha_{n-1}),A(\alpha_{n-1}),A(\alpha_n)}(t)) \ldots\ldots\ldots\ldots\ldots\text{(4.2)}$$

Setting $\theta_n(t) = S_{(A(\alpha_n),A(\alpha_n),A(\alpha_{n+1})}(t)$ then,

$\phi(\theta_n(t)) \le \phi(\theta_{n-1}(t)) - \Psi(\theta_{n-1}(t))$

Since ϕ is strictly decreasing, it is easy to show that $\{(\theta_n(t)\}$ is an increasing sequence for every $t > 0$ with respect to n. That is $\theta_n(t) \ge \theta_{n-1}(t)$ for all $n \ge 1$. We put $\lim_{n\to\infty} \theta_n(t) = \theta(t)$ and assume that $0 < \theta(t) < 1$. From (4.2) , we have

$\phi(\theta_n(t)) \le \phi(\theta_{n-1}(t)) - \Psi(\theta_{n-1}(t))$ ……… (4.3)

for every t, by supposing $n \to \infty$,Since ϕ is left continuous, we have

$\phi(\theta(t)) \le \phi(\theta(t)) - \Psi(\theta(t))$ ……………. (4.4)

which implies that $\Psi(\theta(t)) = 0$. Hence $\theta(t) = 1$. That is the sequence $\{(\theta_n(t)\}$ converges to 1 for any $t > 0$. Next, we show that the sequence $\{A\alpha_n\}$ is a Cauchy sequence. Assume that it is not, then there exist $0 < \varepsilon < 1$ and two sequence $\{p(n)\}$ and $\{q(n)\}$ such that for every $n \in N\cup\{0\}$ and $t > 0$, we obtain:

$$p(n) > q(n) \ge n$$

$S_{(\alpha_{p(n)},\alpha_{p(n)},\alpha_{q(n)})}(t) \le 1 - \varepsilon$ ………… (4.5)

$$S_{(A\alpha_{p(n)-1},A\alpha_{p(n)-1},A\alpha_{q(n)-1})}(t) > 1 - \varepsilon$$

$$S_{(A\alpha_{p(n)-1},A\alpha_{p(n)-1},A\alpha_{q(n)})}(t) > 1 - \varepsilon$$

for each $n \in N\cup\{0\}$, we assume that $\delta_n(t) = S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{q(n)})}(t)$, then we have

$$1 - \varepsilon \ge \delta_n(t) = S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{q(n)})}(t)$$

$$> S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{p(n)-1})}\left(\frac{t}{3}\right)$$

$$* S_{(A\alpha_{p(n)},A\alpha_{p(n)},A\alpha_{p(n)-1})}\left(\frac{t}{3}\right)$$

$$* S_{(A\alpha_{q(n)},A\alpha_{q(n)},A\alpha_{p(n)-1})}\left(\frac{t}{3}\right)$$

$> \theta_{p(n)-1}\left(\frac{t}{3}\right) * \theta_{p(n)-1}\left(\frac{t}{3}\right) * 1 - \varepsilon$ [by (4.5)] ………………. (4.6)

Since $\theta_{p(n)-1}\left(\frac{t}{3}\right) \to 1$ as $n \to \infty$ for every t. We note that $\{\delta_n(t)\}$ converges to $1 - \varepsilon$ as $n \to \infty$ for any $t > 0$, moreover by (4.1), we have

$$\phi(\delta_n(t)) \le \phi(\delta_{n-1}(t)) - \Psi(\delta_{n-1}(t)) \ldots\ldots\ldots. (4.7)$$

Going to the limit in (4.7) as$n \to \infty$, for every $t > 0$, we obtain:

$$\phi(1-\varepsilon) \le \phi(1-\varepsilon) - \Psi(1-\varepsilon)$$

Clearly, this leads to $1 - \varepsilon = 1$, which is a contradiction. Hence$\{A\alpha_n\}$is a Cauchy sequence in the complete S-Menger space X. Therefore we conclude that there exists a point $\beta \in X$ such that $\lim_{n\to\infty} A\alpha_n = \lim_{n\to\infty} B\alpha_{n-1} = \beta$.

It can be seen that from (4.1) and the properties of ϕ and Ψ, that the continuity of A implies the continuity of B .So $B(A\alpha_n) \to B(\beta)$. However $B(A\alpha_n) = A((B\alpha_n)$ by the commutativity of A and B. So $A(B\alpha_n)$converges to$A(\beta)$. Because the limit is unique $A(\beta) = B(\beta)$. So by commutativity, we have $A(A(\beta)) = A(B(\beta))$ and

$$\phi(S_{(B\beta,B\beta,B(B(\beta))}(t) = \phi(S_{(A\beta,A\beta,A(B(\beta))}(t)$$

$$\le \phi(S_{(B\beta,B\beta,B(B(\beta))}(t) - \Psi(S_{(B\beta,B\beta,B(B(\beta))}(t)$$

Hence, necessarily $B(\beta) = B(B(\beta))$,thus $B(\beta) = B\big(B(\beta)\big) = A(B(\beta))$. So $B(\beta)$ is a common fixed point of A and B. Now we prove the uniqueness of the common fixed point of A and B.

If β and γ are two common fixed points to A and B with $\ne \gamma$, then

$$\phi(S_{(\beta,\beta,\gamma)}(t) = \phi(S_{(B\beta,B\beta,B\gamma)}(t)$$

$$\le \phi(S_{(A\beta,A\beta,A\gamma)}(t) - \Psi(S_{(A\beta,A\beta,A\gamma)}(t)$$

$$\phi(S_{(\beta,\beta,\gamma)}(t) - \Psi(S_{(\beta,\beta,\gamma)}(t)$$

Then $\Psi(S_{(\beta,\beta,\gamma)}(t)) \le 0$, so $(S_{(\beta,\beta,\gamma)}(t) = 1$ contradiction.

Example 4.2. Let $X = [0,\infty)$, $a * b * c = a.b.c$ for all a, b, $c\in$ [0, 1]. Define $S: X \times X \times X \to [0,\infty) \to [0,1]$ by

$$S_{(\alpha,\beta,\gamma)}(t) = \begin{cases} 0, & \text{if t} = 0 \text{, for all} \alpha,\beta,\gamma \in X \\ e^{\frac{-[|\alpha-\beta|+|\beta-\gamma|+|\gamma-\alpha|]}{2t}}, & \text{if} t > 0\,. \end{cases}$$

We claim that $(X,S,*)$ is an S- Menger space. In fact, it is enough to prove that for $r,s,t > 0$, $\alpha,\beta,\gamma,a \in X$

$$S_{(\alpha,\alpha,a)}(r) * S_{(\beta,\beta,a)}(s) * S_{(\gamma,\gamma,a)}(t)$$

$$\le e^{\frac{-2|\alpha-a|}{2r}}.e^{\frac{-2|\beta-a|}{2s}}.e^{\frac{-2|\gamma-a|}{2t}}$$

$$\le e^{\frac{-2|\alpha-a|}{2(r+s+t)}}.e^{\frac{-2|\beta-a|}{2(r+s+t)}}.e^{\frac{-2|\gamma-a|}{2(r+s+t)}}$$

$$= e^{\frac{-2[|\alpha-a|+|\beta-a|+|\gamma-a|]}{2(r+s+t)}}$$

$$= e^{\frac{-[|\alpha-\beta|+|\beta-\gamma|+|\gamma-\alpha|]}{(r+s+t)}}$$

$$= S_{(\alpha,\beta,\gamma)}(r+s+t)$$

$\phi = 1 - \sqrt{t}$, $\Psi = 1 - \frac{\sqrt{t}}{2}$, for $t \in [0,1]$

$f(\alpha) = \frac{\alpha}{2}$ and $g(\alpha) = \frac{\alpha}{6}$, for $\alpha \in X$.

Application: Let $Y = \{\,\Omega: [0,1[\to [0,1[\,,\Omega$ is a Lebesgue integrable mapping which is summable, non-negative and satisfies $\int_{1-\varepsilon}^{1} \Omega(\text{t})dt > 0$ for each $0 < \varepsilon < 1\}$

Theorem 4.3. Let $(X,S,*)$ be a complete S-Menger Spaces and assume that $\phi,\Psi: [0,1] \to [0,1]$ satisfy the foregoing properties (q_1), (q_2) and (q_3).Let A

and B be maps that satisfy the following condition:
$(i)\ B(X) \subset A(X)$
$(ii) A$ is continuous and

$$\int_{1-\phi(S_{(B\alpha,B\alpha,B\beta)}(t))}^{1} \Omega(s)ds \leq \int_{1-\phi(S_{(A\alpha,A\alpha,A\beta)}(t)}^{1} \Omega(s)ds - \int_{1-\Psi((S_{(A\alpha,A\alpha,A\beta)}(t))}^{1} \Omega(s)ds \ldots (4.3)$$

for $\beta \in \Omega$, where $\alpha, \beta \in X$,and $\alpha \neq \beta$.Then A and B have a unique common fixed point.

Proof. For $\in \Omega$, we consider the function $\wedge: [0,1] \rightarrow [0,1]$ by $\wedge(\varepsilon) = \int_{1-\varepsilon}^{1} \Omega(s)ds$.

$\wedge$ is continuous, $\wedge(0) = 0, \wedge$ is strictly increasing (4.3) becomes.

$$\wedge\left(\phi\left(S_{(B\alpha,B\alpha,B\beta)}(t)\right)\right) = \wedge\left(\phi\left(S_{(A\alpha,A\alpha,A\beta)}(t)\right) - \wedge\left(\Psi\left(S_{(A\alpha,A\alpha,A\beta)}(t)\right)\right.\right.$$

Setting $\phi_1 = \wedge o\phi$ and $\Psi_1 = \wedge o\Psi_1$. ϕ_1 is strictly decreasing,continuous and satisfies the foregoing properties (q_1) and (q_2) for any $t > 0$, and Ψ_1 satisfies the property(q_3) then by theorem(4.1)A and B have a unique common fixed point.

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